Sheet 1

Exercise 1.1

Part 1
For any $n \in \mathbb{N}$, we set $f_n := \mathbf{1}_{[n,n+1]}$.

1. Show that for any $x \in \mathbb{R}_+$, $\lim_{n\to+\infty} f_n(x) = 0$
2. Show that for any $n \in \mathbb{N}$, we have $\int_{\mathbb{R}_+} f_n(x)dx = 1$

Part 2
We will show that the sequence $(f_n)_{n\in\mathbb{N}}$ does not satisfy the following property: there exist a non-negative function $g \in L^1(\mathbb{R}_+)$ such that

$$\text{a.e. } x \in \mathbb{R}_+,\ \forall n \in \mathbb{N}, \quad |f_n(x)| \leq g(x). \tag{1}$$

1. Show that for any $x \in \mathbb{R}_+$
$$\sup_{n\in\mathbb{N}}\{|f_n(x)|\} = 1.$$
2. Show that, if a measurable function $g : \mathbb{R}_+ \to \mathbb{R}$ satisfying (6), then $g \notin L^1(\mathbb{R}_+)$.

Exercise 1.2 (The Fourier transform of complex Gaussians)

We recall that the Fourier transform $\widehat{f}$ of a function $f \in L^1(\mathbb{R}^d)$ is given for any $\xi \in \mathbb{R}^d$ by the following formula

$$\widehat{f}(\xi) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)dx$$

Let $a \in \mathbb{C}$ such that $\mathrm{Re}(a) > 0$. The goal of this exercise is to show that

$$\forall x \in \mathbb{R}^d, \quad \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-ix\cdot\xi} e^{-\frac{|x|^2}{2a}} dx = a^{\frac{d}{2}} e^{-\frac{a}{2}|\xi|^2} \tag{2}$$

Part 1
For any $x \in \mathbb{R}$, we define $h(x) := e^{-\frac{x^2}{2a}}$. We assume that $h \in \mathscr{S}(\mathbb{R}^d)$.

1. Show that $h'(x) = -\frac{x}{a}h(x)$.
2. Show that $h^{'} \in L^1(\mathbb{R})$ and that $\widehat{h'}(\xi) = i\xi\widehat{h}(\xi)$.

3. Show that $\widehat{h}'(\xi) = -i\widehat{xh}(\xi)$.

4. Recall that
$$\int_{\mathbb{R}} h(x)dx = \sqrt{2a\pi}.$$
Show that $\widehat{h}(0) = \sqrt{a}$.

5. Deduce that $\widehat{h}$ is the solution of the following Cauchy problem
$$\begin{cases} \widehat{h}'(\xi) = -a\xi\widehat{h}(\xi) & \text{in } \mathbb{R}, \\ \widehat{h}(0) = \sqrt{a}. \end{cases} \tag{3}$$

6. Deduce from the Cauchy-Lipschitz theorem that, for any $\xi \in \mathbb{R}$
$$\widehat{h}(\xi) = \sqrt{a}e^{-\frac{a}{2}\xi^2}.$$

Part 2 By remarking that for any $x = (x_1, \dots, x_d) \in \mathbb{R}^d$ we have
$$e^{-\frac{|x|^2}{2a}} = \prod_{j=1}^{d} h(x_j),$$
show Formula (7).

Exercise 1.3 (The heat equation)

Let $u_0 \in \mathscr{S}(\mathbb{R}^d)$. For any $t \geq 0$ and $\xi \in \mathbb{R}^d$, we set
$$u(t,x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t|\xi|^2} \widehat{u}_0(\xi) d\xi.$$

Part 1

1. Show that for any $(t,x) \in (0,+\infty) \times \mathbb{R}^d$, we have
$$\partial_t u(t,x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} (-|\xi|^2) e^{ix\cdot\xi} e^{-t|\xi|^2} \widehat{u}_0(\xi) d\xi.$$

2. Show that $u \in \mathscr{C}^\infty((0,+\infty) \times \mathbb{R}^d)$.

3. Show that $\partial_t u - \Delta u = 0$ in $(0,+\infty) \times \mathbb{R}^d$.

Part 2

1. Show that $(t,x) \in (0,+\infty) \times \mathbb{R}^d$,

$$\frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{-t|\xi|^2} \widehat{u}_0(\xi) d\xi = \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-\frac{|x-y|^2}{4t}} u_0(y) dy.$$

2. Show that $\lim_{t\to 0^+} u(t,x) = u_0(x)$.
3. Deduce that for any $x \in \mathbb{R}^d$, we have $u(0,x) = u_0(x)$.

Part 3
Show that, for any $f \in \mathscr{S}(\mathbb{R}^d)$, we have

$$f(x) = \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} \widehat{f}(\xi) d\xi.$$

Homework (hand in on 22.01.2025).

Exercise 1.4 (The generalised Leibniz rule)

For multiindeces $\alpha, \beta \in \mathbb{N}^d$, we declare that $\beta \leq \alpha$ if $\beta_j \leq \alpha_j$ for all $j = 1, \ldots, d$. Denote by

$$\binom{\alpha}{\beta} = \prod_{j=1}^{d} \binom{\alpha_j}{\beta_j}.$$

Prove the generalised Leibniz formula for $f, g \in \mathscr{C}^{|\alpha|}(\mathbb{R}^d)$

$$\partial^\alpha (fg) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g).$$

Exercise 1.5 (The Schrödinger equation)

Let $u_0 \in \mathscr{S}(\mathbb{R}^d)$. For any $(t,x) \in \mathbb{R} \times \mathbb{R}^d$ we set

$$u(t,x) := \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{ix\cdot\xi} e^{it|\xi|^2} \widehat{u}_0(\xi) d\xi.$$

a) Show that $u \in \mathscr{C}^\infty(\mathbb{R} \times \mathbb{R}^d)$.

b) Show that u solves the Schrödinger equation

$$\begin{cases} \partial_t u + i\Delta u = 0, & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ \lim_{t\to 0} u(t,x) = u_0(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{4}$$

(*Hint:* Use Exercise 1.3 Part 3 to show that $u(0,x) = u_0(x)$.)

Exercise 1.6 (The wave equation)

Let u_0 and u_1 in $\mathscr{S}(\mathbb{R}^d)$. For any $(t,x) \in \mathbb{R}\times\mathbb{R}^d$ we set

$$u(t,x) := \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\cos(t|\xi|)\widehat{u}_0(\xi)d\xi + \frac{1}{(2\pi)^{\frac{d}{2}}}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\frac{\sin(t|\xi|)}{|\xi|}\widehat{u}_1(\xi)d\xi.$$

1. Show that $u \in \mathscr{C}^\infty(\mathbb{R}\times\mathbb{R}^d)$.

2. Show that u solves the wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = 0, & \text{in } \mathbb{R}\times\mathbb{R}^d, \\ \lim_{t\to 0} u(t,x) = u_0(x) \quad \text{and} \quad \lim_{t\to 0}\partial_t u(t,x) = u_1(x), & \text{in } \mathbb{R}^d. \end{cases} \tag{5}$$