Sheet 2

Exercise 1.1

Let f and g in $\mathscr{S}(\mathbb{R}^d)$ and P be a polynomial function. Show the following properties

- $fg \in \mathscr{S}(\mathbb{R}^d)$,
- $Pf \in \mathscr{S}(\mathbb{R}^d)$.

Exercise 1.2 (Transport equation)

Let $u_0 \in \mathscr{S}(\mathbb{R}^d)$ and $v \in \mathbb{R}^d$.

1. Let us set for any $(t,\xi) \in \mathbb{R} \times \mathbb{R}^d$, $\Phi(t,\xi) := e^{itv\cdot\xi}\widehat{u}_0(\xi)$.
   (a) Show that for any $t \in \mathbb{R}$, $\Phi(t,\cdot) \in \mathscr{S}(\mathbb{R}^d)$.
   (b) Show that the function $u : (t,x) \in \mathbb{R}\times\mathbb{R}^d \mapsto u(t,x) := \mathcal{F}^{-1}(\Phi(t,\cdot))(x)$ satisfies
   $$\begin{cases} \partial_t u - v \cdot \nabla u = 0 & \text{in } \mathbb{R} \times \mathbb{R}^d, \\ u(0,\cdot) = u_0 & \text{in } \mathbb{R}^d. \end{cases}$$
2. Using the Fourier inversion formula, find $\varphi : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ such that for any $(t,x) \in \mathbb{R} \times \mathbb{R}^d$, we have $u(t,x) = u_0(\varphi(t,x))$.
3. Let $p \in [1,+\infty]$. Show that
$$\forall t \in \mathbb{R}, \quad \|u(t,\cdot)\|_{L^p} = \|u_0\|_{L^p}.$$

Exercise 1.3 (Generalized Hölder estimate)

Let p, q and r in $[1,\infty]$ such that
$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}.$$
Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$. The goal of this exercise is to show that
$$\|fg\|_{L^r} \le \|f\|_{L^p}\|g\|_{L^q}. \tag{1}$$

1. Show (1) for $r = \infty$.
2. Assume that $r \neq \infty$. Deduce (1) from the standard Hölder estimate (which correspond to the case $r = 1$).
   Hint: use that $r/p + r/q = 1$.

Exercise 1.4 (Young estimate)

Let p, q and r in $[1, \infty]$ such that
$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}.$$
Let $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$. The goal of this exercise is to show that $f \star g \in L^r(\mathbb{R}^d)$, with
$$\|f \star g\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}. \tag{2}$$

1. Assume that $r = \infty$. Show that (2) holds.
2. Assume that $p = q = 1$. Show that (2) holds.
3. Assume that $p = 1$.
   (a) Show that
   $$\left(\int_{\mathbb{R}^d} |f(x-y)||g(y)| dy \right)^q \leq \left(|f| \star |g|^q \right)(x) \|f\|_{L^1}^{q-1}.$$
   Hint: Remark that $|f(x-y)||g(y)| = |f(x-y)|^{1-\frac{1}{q}} |f(x-y)|^{\frac{1}{q}} |g(y)|$.
   (b) Deduce from 2. that
   $$\|f \star g\|_{L^q} \leq \|f\|_{L^1} \|g\|_{L^q}.$$
4. Assume that p, q and r belong to $]1, \infty[$.
   (a) Let p_1, p_2 and p_3 in $[1, \infty]$ such that $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$ and $u \in L^{p_1}(\mathbb{R}^d)$, $v \in L^{p_2}(\mathbb{R}^d)$ and $w \in L^{p_3}(\mathbb{R}^d)$. Show that
   $$\|uvw\|_{L^1} \leq \|u\|_{L^{p_1}} \|v\|_{L^{p_2}} \|w\|_{L^{p_3}}.$$
   (b) Show that $|f(x-y)||g(y)| = |f(x-y)|^{p/r} |g(y)|^{q/r} |f(x-y)|^{1-p/r} |g(y)|^{1-q/r}$.
   (c) Conclude.

Exercise 1.5 (Minkowski estimate)

Let $p \in [1, \infty]$ and $g, f \in L^p(\mathbb{R}^d)$. The goal is to show that
$$\|f + g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}. \tag{3}$$

1. Show (3) for $p = \infty$ and $p = 1$.
2. Assume that $p \in]1, \infty[$.
   (a) Show that
   $$|f(x) + g(x)|^p \leq |f(x)||f(x) + g(x)|^{p-1} + |g(x)||f(x) + g(x)|^{p-1}.$$
   (b) Show that
   $$\int_{\mathbb{R}^d} |f(x)||f(x) + g(x)|^{p-1} dx \leq \|f\|_{L^p} \|f + g\|_{L^p}^{\frac{p-1}{p}}.$$
   (c) Deduce (3).

Exercise 1.6 (Chebyshev estimate)

Let $p \in [1, \infty[$. Show that

$$\forall \lambda > 0, \quad \int_{\mathbb{R}^d} \mathbf{1}_{\{|f| \geq \lambda\}} dx \leq \frac{1}{\lambda^p} \|f\|_{L^p}^p.$$

Homework (hand in on 12.02.2025).

Exercise 1.7 (Interpolation estimate)

Let p and q in $[1, \infty]$ such that $p < q$. Show that if $f \in L^p(\mathbb{R}^d) \cap L^q(\mathbb{R}^d)$, then $f \in L^r(\mathbb{R}^d)$ for every $r \in [p, q]$.
Hint: Use that if $r \in [p, q]$, then there exists $\theta \in [0, 1]$ such that $1/r = \theta/p + (1-\theta)/q$ and show that $\|f\|_{L^r} \leq \|f\|_{L^p}^\theta \|f\|_{L^q}^{1-\theta}$.

Exercise 1.8

Let f, g and h in $\mathscr{S}(\mathbb{R}^d)$. Show the following properties

- $f \star g = g \star f$,
- $f \star (g + h) = f \star g + f \star h$,
- $(f \star g) \star h = f \star (g \star h)$.

Exercise 1.9 (Wave equation and finite propagation speed)

Let us consider a real valued function $u : [0, +\infty[\times \mathbb{R}^d \to \mathbb{R}$. solution of the wave equation.

$$\partial_t^2 u - \Delta u = 0 \quad \text{in }]0, +\infty[\times \mathbb{R}^d.$$

Assume that

(H1) $u \in \mathscr{C}_b^2([0, +\infty[\times \mathbb{R}^d)$;

(H2) there exists $R > 0$, such that $u(0, \cdot)$ and $\partial_t u(0, \cdot)$ vanish on $B(0, R) := \{x \in \mathbb{R}^d \mid |x| \leq R\}$.

The goal of this exercise is to show that

$$u = 0 \text{ in } K(R) := \{(t, x) \in [0, +\infty[\times \mathbb{R}^d \mid |x| \leq R - t\}.$$

Part 1

For any $\varepsilon \geq 0$ and $(t, x) \in [0, +\infty[\times \mathbb{R}^d$, we set

$$\varphi_\varepsilon(t, x) := R - (t + \sqrt{|x|^2 + \varepsilon}).$$

1. Sow that for any $t \in [0, +\infty[$ and $s > 0$, the following quantity

$$E_s^\varepsilon(t) := \frac{1}{2} \int_{\mathbb{R}^d} e^{2s\varphi_\varepsilon(t,x)} (|\partial_t u(t,x)|^2 + |\nabla u(t,x)|^2) dx,$$

is well-defined.

2. Assume that $\varepsilon > 0$.

   (a) Show that

$$\frac{d}{dt} E_s^\varepsilon = -s \int_{\mathbb{R}^d} e^{2s\varphi_\varepsilon} (|\partial_t u|^2 + |\nabla u|^2) dx - 2s \int_{\mathbb{R}^d} e^{2s\varphi_\varepsilon} (\nabla \varphi_\varepsilon \cdot \nabla u) \partial_t u dx.$$

   (b) Show that $\|\nabla \varphi_\varepsilon(t, \cdot)\|_{L^\infty} \leq 1$.
   (*Hint*: recall that $\|\nabla \varphi_\varepsilon(t, \cdot)\|_{L^\infty} = \sup_{x \in \mathbb{R}^d} \left(\sum_{j=1}^d |\partial_j \varphi_\varepsilon(t,x)|^2 \right)^{1/2}$).

   (c) Show that

$$-2 \int_{\mathbb{R}^d} e^{2s\varphi_\varepsilon} (\nabla \varphi_\varepsilon \cdot \nabla u) \partial_t u dx \leq \int_{\mathbb{R}^d} e^{2s\varphi_\varepsilon} (|\partial_t u|^2 + |\nabla u|^2) dx.$$

   (*Hint*: use the estimate $2ab \leq a^2 + b^2$)

   (d) Deduce that

$$\forall t \in [0, +\infty[, \quad E_s^\varepsilon(t) \leq E_s^\varepsilon(0).$$

3. Deduce from the dominated convergence theorem that

$$\forall t \in [0, +\infty[, \quad E_s^0(t) \leq E_s^0(0).$$

4. Deduce from 3. that

$$\forall t \in [0, +\infty[, \quad \lim_{s \to +\infty} E_s^0(t) = 0.$$

   (*Hint*: use that $\varphi_0(0, x) < 0$ when $x \in B(0, R)$ and (**H2**)).

5. Conclude that

$$\forall (t, x) \in K(R), \quad u(t, x) = 0.$$