Sheet 4

Exercise 1.1 (Slowly decaying function)

We define the set

$$\mathscr{R}(\mathbb{R}^d) := \left\{ f : \mathbb{R}^d \to \mathbb{C} \text{ measurable} \;\middle|\; \exists a \in \mathbb{N},\ \langle \cdot \rangle^{-a} f \in L^1(\mathbb{R}^d) \right\},$$

where $\langle \cdot \rangle := (1+|\cdot|^2)^{\frac{1}{2}}$. Sow that any element of $\mathscr{R}(\mathbb{R}^d)$ is a regular distribution.

Exercise 1.2 (Leibniz rule in $\mathscr{S}'$)

Let $\alpha \in \mathbb{N}^d, T \in \mathscr{S}'(\mathbb{R}^d)$ and $f \in \mathscr{S}(\mathbb{R}^d)$. Show that $\partial^\alpha (fT) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\beta - \alpha} T$. (*Hint:* begin the induction by the case $|\alpha| = 1$.)

Exercise 1.3 (Classical distribution)

Show that the following maps are distribution

1. (Dirac distribution) For $a \in \mathbb{R}^d$, $\delta_a : f \in \mathscr{S}(\mathbb{R}^d) \mapsto f(a)$.

Exercise 1.4 (Operation with distribution)

Derivation

1. Show that $(\mathbf{1}_{\mathbb{R}_+})' = \delta_0$ in $\mathscr{S}'(\mathbb{R})$
2. Show that $(\mathrm{sgn})' = 2\delta_0$ in $\mathscr{S}'(\mathbb{R})$

Multiplication

1. Let $T \in \mathscr{S}'(\mathbb{R}^d)$ and f be polynomial function. Show that $fT \in \mathscr{S}'(\mathbb{R}^d)$.

Convolution

1. Compute $\delta_a \star f$, with $a \in \mathbb{R}^d$ and $f \in \mathscr{S}(\mathbb{R}^d)$.

Fourier transform

1. Compute $\mathscr{F}(\delta_a)$ with $a \in \mathbb{R}^d$,
2. Show that $\mathscr{F}(1) = (2\pi)^{\frac{d}{2}} \delta_0$ in $\mathscr{S}'(\mathbb{R}^d)$,

Convergence

1. Show that $\lim_{t \to 0^+} h_t = \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$.

Homework (hand in on 05.03.2025).

Exercise 1.5 (Principal value)

1. (Principal value of $1/x$) Show that $\mathrm{vp}(\frac{1}{x}) : f \in \mathscr{S}(\mathbb{R}) \mapsto \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{f(x)}{x} dx$ is a tempered distribution.
2. Show that $\log(|\cdot|) \in \mathscr{S}'(\mathbb{R})$ and that $(\log(|x|))' = \mathrm{vp}(\frac{1}{x})$ in $\mathscr{S}'(\mathbb{R})$.
3. Show that $x\mathrm{vp}(\frac{1}{x}) = 1$ in $\mathscr{S}'(\mathbb{R})$.
4. Show that $\mathscr{F}(\mathrm{vp}(\frac{1}{x})) = i\sqrt{2\pi}\,\mathbf{1}_{\mathbb{R}_+}$ in $\mathscr{S}'(\mathbb{R})$. (*Hint*: use that $x\mathrm{vp}(\frac{1}{x}) = 1$, $\mathscr{F}(xT) = -i\,\mathscr{F}(T)'$ and $(\mathbf{1}_{\mathbb{R}_+})' = \delta_0$)

Exercise 1.6 (Wigner measure)

We define the Wigner transform at scale $h > 0$ of a function $f \in L^2(\mathbb{R}^d)$ by the following formula

$$\forall (x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d, \quad W^h[f](x,\xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{iy\cdot\xi} f\left(x - \frac{h}{2}y\right) \overline{f}\left(x + \frac{h}{2}y\right) dy.$$

Part 1 (Wigner transform)
Let $f \in L^2(\mathbb{R}^d)$ and $h > 0$.

1. Show that for any $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$, we have
$$W^h[f](x,\xi) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} e^{i\frac{y}{h}\cdot\xi} f\left(x - \frac{y}{2}\right) \overline{f}\left(x + \frac{y}{2}\right) dy.$$
2. Show that $W^h[f] \in L^\infty(\mathbb{R}^{2d})$ and that
$$\|W^h[f]\|_{L^\infty(\mathbb{R}^{2d})} \leq \frac{2^d}{h^d} \|f\|^2_{L^2(\mathbb{R}^d)}.$$
3. Show that
$$\|W^h[f]\|_{L^2(\mathbb{R}^{2d})} = \frac{1}{(2\pi h)^{\frac{d}{2}}} \|f\|^2_{L^2(\mathbb{R}^d)}$$
4. Show that
$$\int_{\mathbb{R}^d} W^h[f](x,\xi) d\xi = |f(x)|^2 \quad \text{and} \quad \int_{\mathbb{R}^d} W^h[f](x,\xi) dx = |\widehat{f}\left(\frac{\xi}{h}\right)|^2$$
5. Deduce that
$$\int_{\mathbb{R}^{2d}} W^h[f](x,\xi) d\xi dx = \|f\|^2_{L^2(\mathbb{R}^d)}.$$

Part 2 (Wigner distribution)

1. Let $f \in L^2(\mathbb{R}^d)$ and $h > 0$. Show that the maps $a \in \mathscr{S}(\mathbb{R}^{2d}) \mapsto \int_{\mathbb{R}^{2d}} a(x,\xi) W^h[f](x,\xi) d\xi dx$ define a tempered distribution on $\mathbb{R}^{2d}$. **In the following we denote also by $W^h[f]$ this distribution.**

Part 3 (Wigner measure)
Let $(f_h)_{h>0}$ be a bounded family of $L^2(\mathbb{R}^{2d})$. We say that $(f_h)_{h>0}$ admit a *Wigner measure* T, if $T \in \mathscr{S}'(\mathbb{R}^{2d})$ and if for all sequence of positive real numbers $(h_n)_{n\in\mathbb{N}}$ converging to 0, $W^{h_n}[f_{h_n}]$ converge to T in $\mathcal{S}'(\mathbb{R}^{2d})$.

1. Let $\psi \in \mathscr{C}_c(\mathbb{R}^d)$. For any $a \in \mathscr{S}(\mathbb{R}^{2d})$, we define $T_\psi(a) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} a(x,0)\psi(x)dx$. Show that $T_\psi \in \mathscr{S}'(\mathbb{R}^d)$.
2. (Wigner measure of traveling wave) Let $\alpha \in]0,1[$, $k \in \mathbb{R}^d$ and $f \in \mathscr{C}_c(\mathbb{R}^d)$. For any $h > 0$ and $x \in \mathbb{R}^d$, we set $f_h(x) := f(x) e^{\frac{i}{h^\alpha} k\cdot x}$. Show that $T_{|f|^2}$ is a Wigner measure of $(f_h)_{h>0}$.
3. (Wigner measure of Coherent states) For any $(x_0,\xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ and $h > 0$, we define the function $\Psi_h^{\xi_0,x_0}$ by setting

$$\forall x \in \mathbb{R}^d, \quad \Psi_h^{\xi_0,x_0}(x) := \frac{1}{(\pi h)^{\frac{d}{4}}} e^{-\frac{|x-x_0|^2}{2h}} e^{\frac{i}{h}\xi_0\cdot x}.$$

   (a) Show that

$$W^h[\Psi_h^{\xi_0,x_0}](x,\xi) = \frac{1}{(\pi h)^d} e^{\frac{|x-x_0|^2+|\xi-\xi_0|^2}{h}}.$$

   (b) Show that $\delta_{(x_0,\xi_0)}$ is a Wigner measure of $(\Psi_h^{\xi_0,x_0})_{h>0}$.

Part 4 (From the Schrödinger equation to the transport equation)

1. Let $h > 0$ and $f \in \mathscr{C}^\infty(\mathbb{R}\times\mathbb{R}^d) \cap \mathscr{C}^1(\mathbb{R};\mathscr{S}(\mathbb{R}^d))$ be a solution of the Schrödinger equation

$$\partial_t f - \frac{h}{2} i\Delta f = 0 \ \text{ in } \mathbb{R}\times\mathbb{R}^d.$$

   We admit that $W^h[f(t,\cdot)] \in \mathscr{S}(\mathbb{R}^{2d})$ for any $t \in \mathbb{R}$. Show that, for any $\xi \in \mathbb{R}^d$, the function $\rho : (t,x) \in \mathbb{R}\times\mathbb{R}^d \mapsto W^h[f(t,\cdot)](x,\xi)$ is a solution of the transport equation

$$\partial_t \rho + \xi\cdot\nabla\rho = 0 \ \text{ in } \mathbb{R}\times\mathbb{R}^d.$$