

Sheet 5

Exercise 1.1 (The Shrödinger equation and uniqueness)

For any $t \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^d$, we set

$$k_t(x) := \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{\frac{ix|^2}{4t}}.$$

1. Let $u_0 \in \mathcal{S}'(\mathbb{R}^d)$. Show that the maps $t \in \mathbb{R} \mapsto k_t \star u_0$ belongs to $C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^d))$ and define a temperate distribution on $\mathbb{R} \times \mathbb{R}^d$. (*Hint*: show that for any $f \in \mathcal{S}(\mathbb{R}^d)$, $t \in \mathbb{R} \mapsto k_t \star f$ belongs to $C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^d))$ and extends this result by duality.)

In the following, we set $u := k \star u_0$.

2. Show that u satisfies the Schrödinger equation in $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$, that is

$$(i\partial_t + \Delta)u = 0 \text{ in } \mathcal{S}'(\mathbb{R} \times \mathbb{R}^d).$$

3. Show that, for any $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ and $t \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned} \int_0^t \langle u, (i\partial_t + \Delta)\varphi(s, \cdot) \rangle_{\mathcal{S}'(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d)} ds \\ = \langle u_0, i\varphi(0, \cdot) \rangle_{\mathcal{S}'(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d)} - \langle u(t), i\varphi(t, \cdot) \rangle_{\mathcal{S}'(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d)} \end{aligned} \quad (1)$$

4. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$, $T > 0$ and $\chi_T \in C_c^\infty(\mathbb{R})$ such that $\chi_T(t) = 1$ for any $t \in [0, T]$. We define the function Φ by setting for any $(t, \xi) \in \mathbb{R} \times \mathbb{R}^d$, $\Phi^T(t, \xi) := e^{i(T-t)|\xi|^2} \widehat{\psi}(\xi) \chi_T(t)$.

- (a) Show that $\varphi^T : (t, x) \in \mathbb{R} \times \mathbb{R}^d \mapsto \mathcal{F}^{-1}(\Phi^T(t, \xi))$ belongs to $\mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ and satisfies

$$i\partial_t \varphi^T + \Delta \varphi^T = 0 \text{ in } [0, T] \times \mathbb{R}^d.$$

- (b) Assume that $u_0 = 0$. Deduce from (1) that $u(T) = 0$ in $\mathcal{S}'(\mathbb{R}^d)$.

5. Deduce that for any $u_0 \in \mathcal{S}'(\mathbb{R}^d)$, the distribution $k \star u_0$ is the unique solution of the Schrödinger equation in $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$ satisfying (1) for any $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ and $t \in \mathbb{R} \setminus \{0\}$.

Exercise 1.2 (Green's function for the Laplacian)

1. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be the function given by $g(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$. Show that g (more precisely the associated distribution, φ_g) is the unique solution in $\mathcal{S}'(\mathbb{R})$ to the equation

$$(1 - \Delta)\varphi = \delta_0,$$

by

- (a) the Fourier transform;
 - (b) using the distributional derivative.
2. Prove that for $f \in \mathcal{S}(\mathbb{R})$ the unique solution to the equation

$$(1 - \Delta)u = f$$

is

$$u(x) = \int g(x - y)f(y)dy.$$

Remark 1.2.1. *The function g is called the fundamental solution or Green's function for the equation.*

Exercise 1.3 (Example of distributions belonging in $H^s(\mathbb{R}^d)$)

1. Let $a \in \mathbb{R}^d$. Show that $\delta_a \in H^{-s}(\mathbb{R}^d)$ for any $s > d/2$.
2. Let $a, b \in \mathbb{R}$ such that $a < b$. Show that $\mathbf{1}_{[a,b]} \in H^s(\mathbb{R})$ for any $s < \frac{1}{2}$.
3. Why $\mathbf{1}_{[a,b]} \notin H^s(\mathbb{R})$ for $s > \frac{1}{2}$?

Homework (hand in on 12.03.2025).

Exercise 1.4 (Dispersive estimate for Schrödinger equation)

Let $u_0 \in L^p(\mathbb{R}^d)$ for $p \in [1, 2]$ and u be the solution of the Schrödinger equation in the sens of Exercise 1.1, Item 5. with initial data u_0 .

1. Show that, if $u_0 \in L^1(\mathbb{R}^d)$, then

$$\forall t \in \mathbb{R} \setminus \{0\}, \quad \|u(t)\|_{L^\infty} \leq \frac{1}{(4\pi t)^{\frac{d}{2}}} \|u_0\|_{L^1}$$

2. Show that, if $u_0 \in L^2(\mathbb{R}^d)$, then

$$\forall t \in \mathbb{R} \setminus \{0\}, \quad \|u(t)\|_{L^2} = \|u_0\|_{L^2}.$$

(Hint: use that $\mathcal{F}(k_t) = e^{-it|\cdot|^2}$ in $\mathcal{S}'(\mathbb{R}^d)$.)

3. (Bonus) Let p' a real number such that $\frac{1}{p} + \frac{1}{p'} = 1$. Show that,

$$\forall t \in \mathbb{R} \setminus \{0\}, \quad \|u(t)\|_{L^{p'}} \leq \frac{1}{(4\pi|t|)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{2})}} \|u_0\|_{L^p}.$$

(Hint: use the Riesz-Thorin Theorem.)

Remark 1.4.1. *The estimate shown in the last question is called a dispersive estimate. This estimate are the main tools to derive the Strichartz estimate for the Schrödinger equation. These estimate are use to solve a large class of nonlinear Schrödinger equations.*

Exercise 1.5 (Nonlinear heat equation)

Let $d \geq 3$ and $u_0 \in L^d(\mathbb{R}^d)$.

Part 1: Functional setting. We define the set

$$K := \left\{ u \in \mathcal{C}((0, +\infty); L^{2d}(\mathbb{R}^d)) \mid \|u\|_K := \sup_{s>0} \{s^{\frac{1}{4}} \|u(s, \cdot)\|_{L^{2d}}\} := \|u\|_K < +\infty \right\}.$$

1. Show that K is a Banach space.

Part 2: Nonlinear estimates. For any u, v and w in K , we define

$$\forall t \geq 0, \quad \mathcal{T}(u, v, w)(t) := \int_0^t e^{(t-s)\Delta} (u(s, \cdot)v(s, \cdot)w(s, \cdot)) ds.$$

1. In this question we will show that for any u, v and w in K , we have $\mathcal{T}(u, v, w) \in K$ and that there is a positive constant C such that

$$\|\mathcal{T}(u, v, w)\|_K \leq C_1 \|u\|_K \|v\|_K \|w\|_K. \quad (2)$$

Let u, v and w in K .

- (a) Show that

$$\forall t \geq 0, \quad \|\mathcal{T}(u, v, w)(t)\|_{L^{2d}} \leq \left(\int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} ds \right) \|u(s, \cdot)v(s, \cdot)w(s, \cdot)\|_{L^{\frac{2d}{3}}}.$$

- (b) Deduce that

$$\forall t \geq 0, \quad \|\mathcal{T}(u, v, w)(t)\|_{L^{2d}} \leq \left(\int_0^t \frac{1}{(t-s)^{\frac{1}{2}} s^{\frac{3}{4}}} ds \right) \|u\|_K \|v\|_K \|w\|_K.$$

- (c) Deduce that (2) holds.
- (d) Show that $\mathcal{T}(u, v, w) \in K$.

Part 3: Duhamel formula. Let $u_0 \in L^d(\mathbb{R}^d)$. For any $t \geq 0$ and $u \in K$, we define

$$\Phi(u)(t) := e^{t\Delta} u_0 + \mathcal{T}(u, u, u)(t).$$

1. Show that there exists a constant C_2 such that for any $u \in K$, we have

$$\|\Phi(u)\|_K \leq C_2 (\|u_0\|_{L^d} + \|u\|_K^3) \quad (3)$$

and that $\Phi \in K$.

2. Show that there exists a constant C_3 such that for any u and v in K , we have

$$\|\Phi(u) - \Phi(v)\|_K \leq C_3 (\|u\|_K^2 + \|v\|_K^2 + \|u\|_K \|v\|_K) \|u - v\|_K. \quad (4)$$

(Hint: use Estimate (2))

Part 4: Fixed point argument Let $\varepsilon > 0$. Assume that

$$\|u_0\|_{L^d} < \varepsilon. \quad (5)$$

and we introduce the set

$$B(2\varepsilon) := \{u \in K \mid \|u\|_K < 2\varepsilon\}.$$

1. Show that $B(2\varepsilon)$ is a complete metric space for the distance induced by K norm, namely $\|\cdot\|_K$.
2. Show that there exists a constant C_4 which does not depend on ε , such that for any u and v in $B(2\varepsilon)$, we have

$$\|\Phi(u)\|_K \leq C_4(1 + \varepsilon^2)\varepsilon$$

and

$$\|\Phi(u) - \Phi(v)\|_K \leq C_4\varepsilon^2\|u - v\|_K.$$

3. Choose $\varepsilon > 0$ small enough such that Φ is a strict contraction of $B(2\varepsilon)$.
4. Deduce that Φ has a unique fixed point u . (*Hint*: use the fixed point theorem.)

Remark 1.5.1. *The fixed points of Φ are called the mild solution of the equation*

$$\begin{cases} \partial_t u - \Delta u + u^3 = 0, & \text{in }]0, +\infty[\times \mathbb{R}^d, \\ u(0, \cdot) = u_0, & \text{in } \mathbb{R}^d, \end{cases} \quad (6)$$

for small enough initial data u_0 . We can obtain the existence of mild solution for u_0 large, but only for small times. The same method (so called "Kato method") can be used to show the existence of solution to the Navier-Stokes equation.