Sheet 7

Exercise 1.1 (Spectrum of bounded operator)

Let $\mathcal{H}$ be an Hilbert space and $A \in \mathrm{B}(\mathcal{H})$.

1. Show that, if $\|A\|_{\mathrm{B}(\mathcal{H})} < 1$ is bounded, then $1 + A$ is invertible.
2. Show that $\sigma(A)$ is compact.

Homework (hand in on 02.04.2025).

Exercise 1.2 (The Lax-Milgram Theorem)

Let $\mathcal{H}$ be a Hilbert space and

$$\alpha : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$$

a sesquilinear form. Asssume that

- α is *bounded*: there exists $C > 0$ so that for all $f, g \in \mathcal{H}$
$$|\alpha(f, g)| \leq C\|f\|\|g\|;$$
- α is *coercive*: there exists $a > 0$ so that for all $f \in \mathcal{H}$
$$\alpha(f, f) \geq a\|f\|^2.$$

Prove that:

1. There exists $A \in \mathrm{B}(\mathcal{H})$ so that $\alpha(f, g) = \langle Af, g\rangle$;
2. A is bijective with bounded inverse satisfying $\|A^{-1}\| \leq a^{-1}$;
3. $g = A^{-1}f$ is the unique minimiser of
$$g \mapsto \alpha(g, g) - 2\mathrm{Re}\langle f, g\rangle.$$

Exercise 1.3

Let $V \in L^\infty(\mathbb{R}^d, \mathbb{R})$ be non-negative.

1. Prove that for every $f \in L^2(\mathbb{R}^d)$ and $\lambda > 0$ there exists a unique $u \in H^1(\mathbb{R}^d)$ such that
$$\forall \varphi \in H^1(\mathbb{R}^d) : \langle \nabla u, \nabla \varphi\rangle + \langle (V + \lambda)u, \varphi\rangle = \langle f, \varphi\rangle,$$
that is, there is a unique weak solution to the equation
$$-\Delta u + Vu + \lambda u = f.$$
2. Prove that the weak solution $u \in H^1(\mathbb{R}^d)$ obtained in part 1) is an element of $H^2(\mathbb{R}^d)$.