Sheet 8

Exercise 1.1 (Multiplication operators)

For a (possibly unbounded) measurable function $\varphi : \mathbb{R}^d \to \mathbb{C}$ consider the linear map M_φ in $L^2(\mathbb{R}^d)$ defined by

$$\begin{aligned} \mathcal{D}(M_\varphi) &:= \left\{ f \in L^2(\mathbb{R}^d) \,\middle|\, \varphi f \in L^2(\mathbb{R}^d) \right\} \\ (M_\varphi f)(x) &:= \varphi(x) f(x) \,. \end{aligned}$$

Part 1: General properties

1. Show that $\mathcal{D}(M_\varphi)$ is dense in $L^2(\mathbb{R}^d)$.
2. Show that $(M_\varphi)^* = M_{\overline{\varphi}}$.
3. Show that M_φ is closed.
4. Show that the following property: If $\varphi \in L^\infty(\mathbb{R}^d)$ then M_φ is bounded, and
$$\|M_\varphi\| = \|\varphi\|_\infty = \sup \Big\{ t : \big| \{ x \in \mathbb{R}^d : |\varphi(x)| \geq t \} \big| > 0 \Big\},$$
where $|V|$ denotes the Lebesgue measure of a measurable subset $V \subset \mathbb{R}^d$.

Exercise 1.2 (Cauchy-Lipschitz Theorem)

Let X be a Banach space and $F : X \to X$ be a Lipschitz map.

1. Show that for any $x_0 \in X$, there exists $x \in C([0, +\infty[; X) \cap C^1(]0, +\infty[; X)$ such that
$$\begin{cases} x' = F(x), & \text{in }]0, +\infty[, \\ x(0) = x_0. \end{cases} \tag{1}$$
2. Show that, for a given initial data $x_0 \in X$, this solution is unique.
3. Let $g \in L^\infty(\mathbb{R}^d)$ and $f_0 \in L^2(\mathbb{R}^d)$. Solve the following equation
$$\begin{cases} \partial_t f = g f, & \text{in }]0, +\infty[\times \mathbb{R}^d, \\ f(0, \cdot) = f_0, & \text{in } \mathbb{R}^d, \end{cases} \tag{2}$$
where $f : [0, +\infty[\times \mathbb{R}^d \to \mathbb{C}$.

Exercise 1.3

Let $A \in \mathrm{B}(\mathbb{C}^d) = \mathbb{C}^{d\times d}$ and consider the linear autonomous ODE

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Au(t).$$

Show that

$$\limsup_{t\to\infty} |u(t)| < \infty$$

holds for all solutions if and only if all eigenvalues of A have non-positive real part and the purely imaginary eigenvalues have equal algebraic and geometric multiplicity.

Give examples where the solution exhibits exponential/polynomial growth.

Exercise 1.4 (Transport equation)

Let $v \in \mathbb{R}$ and $A := v \cdot \nabla$ with $D(A) := H^1(\mathbb{R})$.

1. Show that A is maximal dissipative.
2. Show that for $u_0 \in L^2(\mathbb{R})$

$$(\mathrm{e}^{At}u_0)(x) = u_0(x + tv).$$

Exercise 1.5 (Dissipative matrices)

Let $d \in \mathbb{N}$ and $A \in \mathrm{B}(\mathbb{C}^d)$ be a $d \times d$ matrix.

1. Assume there exists a unitary $U \in \mathrm{B}(\mathbb{C}^d)$ so that UAU^* is diagonal and give a necessary and sufficient condition on $\sigma(A)$ for A to be dissipative.
2. Let $d = 2$ and A be the non-trivial Jordan block

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

   Give a necessary and sufficient condition on $\lambda \in \mathbb{C}$ for A to be dissipative.
3. Let A be as in part 2. and $\mathrm{Re}\lambda < 0$. Show that there exists a matrix S such that $B = SAS^{-1}$ is dissipative.

Exercise 1.6 (Ornstein-Uhlenbeck semi-group)

Let us consider $\gamma : x \in \mathbb{R} \mapsto \frac{1}{\sqrt{2\pi}} e^{-\frac{|x|^2}{2}}$ and set

$$L^2(\gamma) := \left\{ f : \mathbb{R} \mapsto \mathbb{R} \text{ measurable } \;\middle|\; \int_{\mathbb{R}} |f(x)|^2 \gamma(x) dx < +\infty \right\}.$$

The for any $t \geq 0$ and $f \in L^2(\gamma)$, we set

$$(\mathrm{U}_t f)(x) := \int_{\mathbb{R}} f(e^{-t}x + \sqrt{1 - e^{-2t}}y)\gamma(y)dy.$$

1. Show that for all $t \geq$ and $f \in L^2(\gamma)$, $\mathrm{U}_t f$ is well-defined.
2. Show that $(\mathrm{U}_t)_{t\geq 0}$ its a contraction semi-group on $L^2(\gamma)$.

   ⋆ In the following, we will denote by A is generator of $(\mathrm{U}_t)_{t\geq 0}$. ⋆

3. Let us set
$$\mathcal{A} := \left\{ f \in C^\infty(\mathbb{R}) \;\middle|\; \forall \alpha \in \mathbb{N}, \exists P \in \mathbb{R}[X] \text{ such that } |f^{(\alpha)}| \leq P \right\}.$$
   Show that $\mathcal{A} \subset D(A)$ and that for any $f \in \mathcal{A}$, we have
$$\forall x \in \mathbb{R}, \quad (Af)(x) = \partial_x^2 f(x) - x\partial_x f(x).$$
4. We admit that $\mathcal{A}$ is stable by $(\lambda - A)^{-1}$ for some $\lambda > 0$. Show that $\overline{\mathcal{A}}^{D(A)} = D(A)$ (*Hint:* use that $\mathcal{A}$ is dense in $L^2(\gamma)$).
5. Show that A is self-adjoint.

Exercise 1.7 (The wave equation)

In this exercise we solve the wave equation on $\mathbb{R}^d$ using the Hille Yosida theorem. The wave equation is
$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(0) = u_0 \\ \partial_t u(0) = v_0. \end{cases} \tag{W}$$

1. Let $\mathcal{H} := H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ and let A be the operator
$$A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$$
   with domain $D(A) := H^2(\mathbb{R}^d) \oplus H^1(\mathbb{R}^d)$. Show that if $(u, v) \in C^1(\mathbb{R}, \mathcal{H})$ is a solution to the Cauchy problem
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(u, v) = A(u, v) \\ (u, v)(0) = (u_0, v_0) \end{cases} \tag{A}$$
   then u solves the wave equation (W).
2. Show that (u, v) solves (A) if and only if $(\tilde{u}, \tilde{v}) = \mathrm{e}^{-t}(u, v)$ solves
$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}(\tilde{u}, \tilde{v}) = (A - 1)(\tilde{u}, \tilde{v}) \\ (\tilde{u}, \tilde{v})(0) = (u_0, v_0). \end{cases}$$
3. Show that $A - 1$ is maximal dissipative.
4. State the existence and uniqueness result for the wave equation implied by 1. and 2. and the Hille-Yosida theorem, specifying the functional space for the solution u.