Sheet 9

Exercise 1.1 (Multiplication operators II)

For a measurable function $\varphi : \mathbb{R}^d \to \mathbb{C}$ consider the linear map M_φ in $L^2(\mathbb{R}^d)$ defined by

$$\begin{aligned} \mathcal{D}(M_\varphi) &:= \left\{ f \in L^2(\mathbb{R}^d) \,\middle|\, \varphi f \in L^2(\mathbb{R}^d) \right\} \\ (M_\varphi f)(x) &:= \varphi(x) f(x) \,. \end{aligned}$$

1. Show that

$$\sigma(M_\varphi) = \{ z \in \mathbb{C} \mid \forall \varepsilon > 0, |\{|z - \varphi| < \varepsilon\}| > 0 \} \,.$$

2. Show that $z \in \mathbb{C}$ is an eigenvalue if and only if

$$|\{\varphi^{-1}(\{z\})\}| > 0.$$

3. Let $\varphi(x) := x \;\; \forall \, x \in \mathbb{R}$. Then the quantum mechanical position operator $q := M_\varphi$ is self-adjoint, has no eigenvalues, and $\sigma(q) = \mathbb{R}$.

Exercise 1.2 (Magnetic Schrödinger operator)

Let $B_1 \in C^1_b(\mathbb{R}^d)^d$ such that $\mathrm{div}(B_1) = 0$. We consider the magnetic Schrödinger operator

$$A := (-i\nabla + B_1)^2, \quad D(A) := H^2(\mathbb{R}^d).$$

Show that $(A, D(A))$ is maximal dissipatif.

Exam preparation (to prepare for the 17.04.2025)

You will have **180 minutes** in total to complete the examination. The following rules will apply:

- Switch off your mobile phone and put it away. Do not leave it on the table.
- You may use the lecture notes of the course in paper form that may include your own annotations. No other material is admitted.
- Do not write with a pencil, nor use the colours red or green.
- All answers and solutions must provide sufficiently detailed arguments, except for multiple choice questions.
- Only one solution to each problem will be accepted. Please cross out everything that is not supposed to count.

Problem 1. *[5 Points]*

Mark all correct statements.

a) Let A, $D(A)$ be the densely defined operator

$$Af := -\frac{\mathrm{d}^2 f}{\mathrm{d}x^2} - \mathrm{i}\frac{\mathrm{d}f}{\mathrm{d}x}$$

with domain $D(A) = H^2(\mathbb{R}) \subset L^2(\mathbb{R}^d) =: \mathcal{H}$. Then A is

- ☐ *dissipative;*
- ☐ *self-adjoint;*

b) The solution operator to the heat equation on $\mathbb{R}^d$, $T(t) := \mathrm{e}^{t\Delta}$, $t \geq 0$,

- ☐ *is bounded on $L^2(\mathbb{R}^d)$;*
- ☐ *has spectrum $\sigma(T(t)) = \{z \in \mathbb{C} : |z| < 1\}$;*
- ☐ *is dissipative.*

Problem 2. *Let $m > 0$, $f \in \mathscr{S}(\mathbb{R})$ and set*

$$u(t,x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \cos(t\sqrt{k^2+m^2}) \mathrm{e}^{-\mathrm{i}kx} \hat{f}(k) \mathrm{d}k.$$

a) *Show that for all $t \in \mathbb{R}$, $(t,x) \mapsto u(t,x) \in C^2(\mathbb{R}^2)$.* *[2 Points]*

b) *Show that u solves the Klein–Gordon equation* *[2 Points]*

$$\partial_t^2 u(t,x) = (\partial_x^2 - m^2) u(t,x)$$

with intial data

$$u(0,x) = f(x), \qquad \partial_t u(0,x) = 0.$$

c) *Show that for all $t \in \mathbb{R}$, $\int |u(t,x)|^2 \mathrm{d}x \leq \|f\|^2_{L^2(\mathbb{R})}$.* *[1 Points]*

Problem 3. *Define for $f \in \mathscr{S}(\mathbb{R})$*

$$\varphi(f) = \frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=0} x f(x).$$

a) *Show that φ defines a tempered distribution.* *[2 Points]*

b) *Calculate the Fourier transform of φ* *[2 Points]*

c) *Show that $\varphi \in H^{-1}(\mathbb{R})$.* *[1 Points]*

Problem 4. *[5 Points]*

Let $a \in C^1(\mathbb{R},\mathbb{R})$ satisfy $a(x) \geq 1$ for all $x \in \mathbb{R}$ and $a, \frac{\mathrm{d}a}{\mathrm{d}x} \in L^\infty(\mathbb{R})$. Prove that for $u_0 \in H^2(\mathbb{R})$ the Cauchy problem

$$\begin{cases} \partial_t u(t,x) = \partial_x a(x) \partial_x u(t,x) + \partial_x u(t,x) \\ \quad u(0) = u_0 \end{cases}$$

admits a unique solution

$$u \in C^1([0,\infty), L^2(\mathbb{R})) \cap C^0([0,\infty), H^2(\mathbb{R})).$$