

Sheet 4

Exercise 1.1 (Slowly decaying function)

We define the set

$$\mathcal{R}(\mathbb{R}^d) := \{f : \mathbb{R}^d \rightarrow \mathbb{C} \text{ measurable} \mid \exists a \in \mathbb{N}, \langle \cdot \rangle^{-a} f \in L^1(\mathbb{R}^d)\},$$

where $\langle \cdot \rangle := (1 + |\cdot|^2)^{\frac{1}{2}}$. Show that any element of $\mathcal{R}(\mathbb{R}^d)$ is a regular distribution.

Exercise 1.2 (Leibniz rule in \mathcal{S}')

Let $\alpha \in \mathbb{N}^d$, $T \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in \mathcal{S}(\mathbb{R}^d)$. Show that $\partial^\alpha(fT) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} T$.
(Hint: begin the induction by the case $|\alpha| = 1$.)

Exercise 1.3 (Classical distribution)

Show that the following maps are distribution

1. (Dirac distribution) For $a \in \mathbb{R}^d$, $\delta_a : f \in \mathcal{S}(\mathbb{R}^d) \mapsto f(a)$.

Exercise 1.4 (Operation with distribution)

Derivation

1. Show that $(\mathbf{1}_{\mathbb{R}_+})' = \delta_0$ in $\mathcal{S}'(\mathbb{R})$
2. Show that $(\text{sgn})' = 2\delta_0$ in $\mathcal{S}'(\mathbb{R})$

Multiplication

1. Let $T \in \mathcal{S}'(\mathbb{R}^d)$ and f be polynomial function. Show that $fT \in \mathcal{S}'(\mathbb{R}^d)$.

Convolution

1. Compute $\delta_a \star f$, with $a \in \mathbb{R}^d$ and $f \in \mathcal{S}(\mathbb{R}^d)$.

Fourier transform

1. Compute $\mathcal{F}(\delta_a)$ with $a \in \mathbb{R}^d$,
2. Show that $\mathcal{F}(1) = (2\pi)^{\frac{d}{2}} \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$,

Convergence

1. Show that $\lim_{t \rightarrow 0^+} h_t = \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$.

Homework (hand in on 18.02.2026).

Exercise 1.5 (Principal value)

- (Principal value of $1/x$) Show that $\text{vp}(\frac{1}{x}) : f \in \mathcal{S}(\mathbb{R}) \mapsto \lim_{\varepsilon \rightarrow 0^+} \int_{\mathbb{R} \setminus [-\varepsilon, \varepsilon]} \frac{f(x)}{x} dx$ is a tempered distribution.
- Show that $\log(|\cdot|) \in \mathcal{S}'(\mathbb{R})$ and that $(\log(|x|))' = \text{vp}(\frac{1}{x})$ in $\mathcal{S}'(\mathbb{R})$.
- Show that $x \text{vp}(\frac{1}{x}) = 1$ in $\mathcal{S}'(\mathbb{R})$.
- Show that $\mathcal{F}(\text{vp}(\frac{1}{x})) = i \frac{\sqrt{\pi}}{\sqrt{2}} \text{sgn}$ in $\mathcal{S}'(\mathbb{R})$.

Exercise 1.6 (Wigner measure)

We define the Wigner transform at scale $h > 0$ of a function $f \in L^2(\mathbb{R}^d)$ by the following formula

$$\forall (x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d, \quad W^h[f](x, \xi) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{iy \cdot \xi} f\left(x - \frac{h}{2}y\right) \bar{f}\left(x + \frac{h}{2}y\right) dy.$$

Part 1 (Wigner transform)

Let $f \in L^2(\mathbb{R}^d)$ and $h > 0$.

- Show that for any $(x, \xi) \in \mathbb{R}^d \times \mathbb{R}^d$, we have

$$W^h[f](x, \xi) = \frac{1}{(2\pi h)^d} \int_{\mathbb{R}^d} e^{i\frac{y}{h} \cdot \xi} f\left(x - \frac{y}{2}\right) \bar{f}\left(x + \frac{y}{2}\right) dy.$$

- Show that $W^h[f] \in L^\infty(\mathbb{R}^{2d})$ and that

$$\|W^h[f]\|_{L^\infty(\mathbb{R}^{2d})} \leq \frac{2^d}{h^d} \|f\|_{L^2(\mathbb{R}^d)}^2.$$

- Show that

$$\|W^h[f]\|_{L^2(\mathbb{R}^{2d})} = \frac{1}{(2\pi h)^{\frac{d}{2}}} \|f\|_{L^2(\mathbb{R}^d)}^2$$

- Show that

$$\int_{\mathbb{R}^d} W^h[f](x, \xi) d\xi = |f(x)|^2 \quad \text{and} \quad \int_{\mathbb{R}^d} W^h[f](x, \xi) dx = |\widehat{f}\left(\frac{\xi}{h}\right)|^2$$

- Deduce that

$$\int_{\mathbb{R}^{2d}} W^h[f](x, \xi) d\xi dx = \|f\|_{L^2(\mathbb{R}^d)}^2.$$

Part 2 (Wigner distribution)

1. Let $f \in L^2(\mathbb{R}^d)$ and $h > 0$. Show that the maps $a \in \mathcal{S}(\mathbb{R}^{2d}) \mapsto \int_{\mathbb{R}^{2d}} a(x, \xi) W^h[f](x, \xi) d\xi dx$ define a tempered distribution on \mathbb{R}^{2d} . **In the following we denote also by $W^h[f]$ this distribution.**

Part 3 (Wigner measure)

Let $(f_h)_{h>0}$ be a bounded family of $L^2(\mathbb{R}^{2d})$. We say that $(f_h)_{h>0}$ admit a *Wigner measure* T , if $T \in \mathcal{S}'(\mathbb{R}^{2d})$ and if for all sequence of positive real numbers $(h_n)_{n \in \mathbb{N}}$ converging to 0, $W^{h_n}[f_{h_n}]$ converge to T in $\mathcal{S}'(\mathbb{R}^{2d})$.

1. Let $\psi \in \mathcal{C}_c(\mathbb{R}^d)$. For any $a \in \mathcal{S}(\mathbb{R}^{2d})$, we define $T_\psi(a) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} a(x, 0) \psi(x) dx$. Show that $T_\psi \in \mathcal{S}'(\mathbb{R}^d)$.
2. (Wigner measure of traveling wave) Let $\alpha \in]0, 1[$, $k \in \mathbb{R}^d$ and $f \in \mathcal{C}_c(\mathbb{R}^d)$. For any $h > 0$ and $x \in \mathbb{R}^d$, we set $f_h(x) := f(x) e^{\frac{i}{h} k \cdot x}$. Show that $T_{|f|^2}$ is a Wigner measure of $(f_h)_{h>0}$.
3. (Wigner measure of Coherent states) For any $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ and $h > 0$, we define the function $\Psi_h^{\xi_0, x_0}$ by setting

$$\forall x \in \mathbb{R}^d, \quad \Psi_h^{\xi_0, x_0}(x) := \frac{1}{(\pi h)^{\frac{d}{4}}} e^{-\frac{|x-x_0|^2}{2h}} e^{\frac{i}{h} \xi_0 \cdot x}.$$

- (a) Show that

$$W^h[\Psi_h^{\xi_0, x_0}](x, \xi) = \frac{1}{(\pi h)^d} e^{-\frac{|x-x_0|^2 + |\xi-\xi_0|^2}{h}}.$$

- (b) What is a Wigner measure of $(\Psi_h^{\xi_0, x_0})_{h>0}$.

Part 4 (From the Schrödinger equation to the transport equation)

1. Let $h > 0$ and $f \in \mathcal{C}^\infty(\mathbb{R} \times \mathbb{R}^d) \cap \mathcal{C}^1(\mathbb{R}; \mathcal{S}(\mathbb{R}^d))$ be a solution of the Schrödinger equation

$$\partial_t f - \frac{h}{2} i \Delta f = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^d.$$

We admit that $W^h[f(t, \cdot)] \in \mathcal{S}(\mathbb{R}^{2d})$ for any $t \in \mathbb{R}$. Show that, for any $\xi \in \mathbb{R}^d$, the function $\rho : (t, x) \in \mathbb{R} \times \mathbb{R}^d \mapsto W^h[f(t, \cdot)](x, \xi)$ is a solution of the transport equation

$$\partial_t \rho + \xi \cdot \nabla \rho = 0 \quad \text{in } \mathbb{R} \times \mathbb{R}^d.$$