

Sheet 5

Exercise 1.1 (Multiplication operators I)

For a (possibly unbounded) measurable function $\varphi : \mathbb{R}^d \rightarrow \mathbb{C}$ consider the linear map M_φ in $L^2(\mathbb{R}^d)$ defined by

$$\begin{aligned} \mathcal{D}(M_\varphi) &:= \{f \in L^2(\mathbb{R}^d) \mid \varphi f \in L^2(\mathbb{R}^d)\} \\ (M_\varphi f)(x) &:= \varphi(x)f(x). \end{aligned}$$

Part 1: General properties

1. Show that $\mathcal{D}(M_\varphi)$ is dense in $L^2(\mathbb{R}^d)$.
2. Show that $(M_\varphi)^* = M_{\bar{\varphi}}$.
3. Show that M_φ is closed.
4. Show that the following property: If $\varphi \in L^\infty(\mathbb{R}^d)$ then M_φ is bounded.

Exercise 1.2 (The Shrödinger equation and uniqueness)

For any $t \in \mathbb{R} \setminus \{0\}$ and $x \in \mathbb{R}^d$, we set

$$k_t(x) := \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{\frac{ix \cdot x}{4t}}.$$

1. Let $u_0 \in \mathcal{S}'(\mathbb{R}^d)$. Show that the maps $t \in \mathbb{R} \mapsto k_t \star u_0$ belongs to $C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^d))$ and define a temperate distribution on $\mathbb{R} \times \mathbb{R}^d$. (*Hint*: show that for any $f \in \mathcal{S}(\mathbb{R}^d)$, $t \in \mathbb{R} \mapsto k_t \star f$ belongs to $C(\mathbb{R}; \mathcal{S}'(\mathbb{R}^d))$ and extends this result by duality.)

In the following, we set $u := k \star u_0$.

2. Show that u satisfies the Schrödinger equation in $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$, that is

$$(i\partial_t + \Delta)u = 0 \text{ in } \mathcal{S}'(\mathbb{R} \times \mathbb{R}^d).$$

3. Show that, for any $\varphi \in \mathcal{S}(\mathbb{R} \times \mathbb{R}^d)$ and $t \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned} \int_0^t \langle u, (i\partial_t + \Delta)\varphi(s, \cdot) \rangle_{\mathcal{S}'(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d)} ds \\ = \langle u_0, i\varphi(0, \cdot) \rangle_{\mathcal{S}'(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d)} - \langle u(t), i\varphi(t, \cdot) \rangle_{\mathcal{S}'(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d)} \quad (1) \end{aligned}$$

4. Let $\psi \in \mathcal{S}(\mathbb{R}^d)$, $T > 0$ and $\chi_T \in C_c^\infty(\mathbb{R})$ such that $\chi_T(t) = 1$ for any $t \in [0, T]$. We define the function Φ by setting for any $(t, \xi) \in \mathbb{R} \times \mathbb{R}^d$, $\Phi^T(t, \xi) := e^{i(T-t)|\xi|^2} \widehat{\psi}(\xi) \chi_T(t)$.

- (a) Show that $\varphi^T : (t, x) \in \mathbb{R} \times \mathbb{R}^d \mapsto \mathcal{F}^{-1}(\Phi^T(t, \xi))$ belongs to $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$ and satisfies

$$i\partial_t \varphi^T + \Delta \varphi^T = 0 \text{ in } [0, T] \times \mathbb{R}^d.$$

- (b) Assume that $u_0 = 0$. Deduce from (1) that $u(T) = 0$ in $\mathcal{S}'(\mathbb{R}^d)$.

5. Deduce that for any $u_0 \in \mathcal{S}'(\mathbb{R}^d)$, the distribution $k \star u_0$ is the unique solution of the Schrödinger equation in $\mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$ satisfying (1) for any $\varphi \in \mathcal{S}'(\mathbb{R} \times \mathbb{R}^d)$ and $t \in \mathbb{R} \setminus \{0\}$.

Exercise 1.3 (Green’s function for the Laplacian)

1. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be the function given by $g(x) = \frac{1}{2}e^{-|x|}$ for $x \in \mathbb{R}$. Show that g (more precisely the associated distribution, φ_g) is the unique solution in $\mathcal{S}'(\mathbb{R})$ to the equation

$$(1 - \Delta)\varphi = \delta_0,$$

by

- (a) the Fourier transform;
- (b) using the distributional derivative.

2. Prove that for $f \in \mathcal{S}'(\mathbb{R})$ the unique solution to the equation

$$(1 - \Delta)u = f$$

is

$$u(x) = \int g(x - y)f(y)dy.$$

Remark 1.3.1. *The function g is called the fundamental solution or Green’s function for the equation.*

Exercise 1.4 (Example of distributions belonging in $H^s(\mathbb{R}^d)$)

- 1. Let $a \in \mathbb{R}^d$. Show that $\delta_a \in H^{-s}(\mathbb{R}^d)$ for any $s > d/2$.
- 2. Let $a, b \in \mathbb{R}$ such that $a < b$. Show that $\mathbf{1}_{[a,b]} \in H^s(\mathbb{R})$ for any $s < \frac{1}{2}$.
- 3. Why $\mathbf{1}_{[a,b]} \notin H^s(\mathbb{R})$ for $s > \frac{1}{2}$?