

## Sheet 6

### Exercise 1.1 (Multiplication operators II)

For a measurable function  $\varphi : \mathbb{R}^d \rightarrow \mathbb{C}$  consider the linear map  $M_\varphi$  in  $L^2(\mathbb{R}^d)$  defined by

$$\begin{aligned} \mathcal{D}(M_\varphi) &:= \{f \in L^2(\mathbb{R}^d) \mid \varphi f \in L^2(\mathbb{R}^d)\} \\ (M_\varphi f)(x) &:= \varphi(x)f(x). \end{aligned}$$

1. Show that

$$\sigma(M_\varphi) = \{z \in \mathbb{C} \mid \forall \varepsilon > 0, \text{Vol}(\{|z - \varphi| < \varepsilon\}) > 0\}.$$

(Hint: in order to show that  $\rho(M_\varphi) \subset \{z \in \mathbb{C} \mid \exists \varepsilon > 0, \text{Vol}(\{|z - \varphi| < \varepsilon\}) = 0\}$ , consider the set  $\{|\varphi - z| < M^{-1}\}$  for some real number  $M > 0$  such that  $M > \|(M_\varphi - z)^{-1}\|_{\mathcal{B}(L^2)}$ .)

2. Show that  $z \in \mathbb{C}$  is an eigenvalue if and only if

$$\text{Vol}(\{\varphi^{-1}(\{z\})\}) > 0.$$

3. Let  $\varphi(x) := x \ \forall x \in \mathbb{R}$ . Then the quantum mechanical position operator  $q := M_\varphi$  is self-adjoint, has no eigenvalues, and  $\sigma(q) = \mathbb{R}$ .

### Exercise 1.2 (Weak and strong convergence)

Let  $\mathcal{H}$  be a Hilbert space and  $(f_n)_{n \in \mathbb{N}}$  a sequence of  $\mathcal{H}$  that converges weakly to  $f$  in  $\mathcal{H}$  and such that  $(\|f_n\|_{\mathcal{H}})_{n \in \mathbb{N}}$  converges to  $\|f\|_{\mathcal{H}}$ . Show that  $(f_n)_{n \in \mathbb{N}}$  converges strongly to  $f$  in  $\mathcal{H}$ .

### Exercise 1.3 (Density in $H^s(\mathbb{R}^d)$ )

Let  $s \in \mathbb{R}$ . Show that  $\mathcal{S}(\mathbb{R}^d)$  is dense in  $H^s(\mathbb{R}^d)$ .

### Exercise 1.4 (Local compact embedding)

Let  $t < s$ ,  $\varphi \in \mathcal{S}(\mathbb{R}^d)$ . The goal of this exercise is to show that the multiplication by  $\varphi$  is a compact operator from  $H^s(\mathbb{R}^d)$  to  $H^t(\mathbb{R}^d)$ .

Let  $(u_n)_{n \in \mathbb{N}}$  be a sequence of  $H^s(\mathbb{R}^d)$  such that  $\sup_{n \in \mathbb{N}} \{\|u_n\|_{H^s}\} \leq 1$ .

1. Show that, up to extraction of a subsequence,  $(u_n)_{n \in \mathbb{N}}$  converging weakly in  $H^s(\mathbb{R}^d)$  to an element  $u$ .  
Let us set  $v_n := u_n - u$ .

2. Show that there exists a constant  $C_1$  such that

$$\sup_{n \in \mathbb{N}} \{\|\varphi v_n\|_{H^s}\} \leq C_1.$$

3. Show that for any positive real number  $R$ , we have

$$\|\varphi v_n\|_{H^t}^2 \leq \int_{B(0,R)} (1 + |\xi|^2)^t |\mathcal{F}(\varphi v_n)|^2 d\xi + \frac{C_1^2}{(1 + R^2)^{s-t}}.$$

Let us consider  $\varepsilon > 0$ .

4. Show that there exists a positive real number  $R$  such that

$$\frac{C_1^2}{(1 + R^2)^{s-t}} < \varepsilon.$$

5. For all  $\xi \in \mathbb{R}^d$ , we set  $\psi_\xi := \mathcal{F}^{-1}((1 + |\cdot|^2)^{-s} \mathcal{F}(\varphi)(\xi - \cdot))$ . Show that, for any  $\xi \in \mathbb{R}^d$ ,  $\psi_\xi$  belongs to  $\mathcal{S}(\mathbb{R}^d)$  and that

$$\forall \xi \in \mathbb{R}^d, \quad \mathcal{F}(\varphi v_n)(\xi) = \langle \psi_\xi, v_n \rangle_{H^s}.$$

6. Deduce that for any  $\xi \in \mathbb{R}^d$ , we have  $\lim_{n \rightarrow +\infty} \mathcal{F}(\varphi v_n)(\xi) = 0$ .

7. Assume that there exists a positive real number  $M > 0$  such that

$$\sup_{\xi \in B(0,R), n \in \mathbb{N}} \{|\mathcal{F}(\varphi v_n)|\} \leq M. \tag{1}$$

8. Conclude.

We will now show (1).

9. Show that there exists a positive real number  $C_2$  such that

$$\forall \mu \in \mathbb{R}^d, \quad |\widehat{\varphi}(\mu)| \leq \frac{C_2}{(1 + |\mu|^2)^{\frac{d}{2} + |s| - 1}}.$$

10. Show that for any  $\xi \in B(0, R)$ ,

$$\begin{aligned} & \int_{\mathbb{R}^d} (1 + |\eta|^2)^{-s} |\widehat{\varphi}(\xi - \eta)|^2 d\eta \\ & \leq C_1 \int_{|\eta| \leq 2R} (1 + |\eta|^2)^{|s|} ds + C_2 \int_{|\eta| \geq 2R} \frac{(1 + |\eta|^2)^{|s|}}{(1 + |\xi - \eta|^2)^{\frac{d}{2} + |s| + 1}} d\eta. \end{aligned}$$

11. Deduce that there exists a positive real number  $C_3$  such that

$$\forall \xi \in B(0, R), \quad \int_{\mathbb{R}^d} (1 + |\eta|^2)^{-s} |\widehat{\varphi}(\xi - \eta)|^2 d\eta \leq C_3 (1 + R^2)^{|s| + \frac{d}{2}}$$

(Hint: to bound  $\int_{|\eta| \geq 2R} (1 + |\eta|^2)^{-s} (1 + |\xi - \eta|^2)^{-(\frac{d}{2} + |s| + 1)} d\eta$ , use that if  $|\xi| \leq R$  and  $|\eta| \geq 2R$ , we have  $|\xi - \eta| \geq \frac{|\eta|}{2}$ .)

12. Deduce that (1) holds.

**Exercise 1.5 (Norm of the heat propagator)**

Let  $t > 0$ . Show that  $\|e^{t\Delta}\|_{B(L^2(\mathbb{R}^d))} = 1$ .

**Homework (hand in on 18.03.2025).****Exercise 1.6 (Dispersive estimate for Schrödinger equation)**

Let  $u_0 \in L^p(\mathbb{R}^d)$  for  $p \in [1, 2]$  and  $u$  be the solution of the Schrödinger equation in the sense of Exercise 1.2, question 5. in Sheet 5 with initial data  $u_0$ .

1. Show that, if  $u_0 \in L^1(\mathbb{R}^d)$ , then

$$\forall t \in \mathbb{R} \setminus \{0\}, \quad \|u(t)\|_{L^\infty} \leq \frac{1}{(4\pi t)^{\frac{d}{2}}} \|u_0\|_{L^1}$$

2. Show that, if  $u_0 \in L^2(\mathbb{R}^d)$ , then

$$\forall t \in \mathbb{R} \setminus \{0\}, \quad \|u(t)\|_{L^2} = \|u_0\|_{L^2}.$$

(Hint: use that  $\mathcal{F}(k_t) = e^{-it|\cdot|^2}$  in  $\mathcal{S}'(\mathbb{R}^d)$ .)

3. (Bonus) Let  $p'$  a real number such that  $\frac{1}{p} + \frac{1}{p'} = 1$ . Show that,

$$\forall t \in \mathbb{R} \setminus \{0\}, \quad \|u(t)\|_{L^{p'}} \leq \frac{1}{(4\pi|t|)^{\frac{d}{2}(\frac{1}{p}-\frac{1}{2})}} \|u_0\|_{L^p}.$$

(Hint: use the Riesz-Thorin Theorem.)

**Remark 1.6.1.** *The estimate shown in the last question is called a dispersive estimate. This estimate are the main tools to derive the Strichartz estimate for the Schrödinger equation. These estimate are use to solve a large class of nonlinear Schrödinger equations.*

**Exercise 1.7 (Nonlinear heat equation)**

Let  $d \geq 3$  and  $u_0 \in L^d(\mathbb{R}^d)$ .

**Part 1: Functional setting.** We define the set

$$K := \left\{ u \in \mathcal{C}((0, +\infty); L^{2d}(\mathbb{R}^d)) \mid \|u\|_K := \sup_{s>0} \{ s^{\frac{1}{4}} \|u(s, \cdot)\|_{L^{2d}} \} := \|u\|_K < +\infty \right\}.$$

1. Show that  $K$  is a Banach space.

**Part 2: Nonlinear estimates.** For any  $u, v$  and  $w$  in  $K$ , we define

$$\forall t \geq 0, \quad \mathcal{T}(u, v, w)(t) := \int_0^t e^{(t-s)\Delta} (u(s, \cdot)v(s, \cdot)w(s, \cdot)) ds.$$

1. In this question we will show that for any  $u, v$  and  $w$  in  $K$ , we have  $\mathcal{T}(u, v, w) \in K$  and that there is a positive constant  $C$  such that

$$\|\mathcal{T}(u, v, w)\|_K \leq C_1 \|u\|_K \|v\|_K \|w\|_K. \quad (2)$$

Let  $u, v$  and  $w$  in  $K$ .

- (a) Show that

$$\forall t \geq 0, \quad \|\mathcal{T}(u, v, w)(t)\|_{L^{2d}} \leq \int_0^t \frac{1}{(t-s)^{\frac{1}{2}}} \|u(s, \cdot)v(s, \cdot)w(s, \cdot)\|_{L^{\frac{2d}{3}}} ds.$$

- (b) Deduce that

$$\forall t \geq 0, \quad \|\mathcal{T}(u, v, w)(t)\|_{L^{2d}} \leq \left( \int_0^t \frac{1}{(t-s)^{\frac{1}{2}} s^{\frac{3}{4}}} ds \right) \|u\|_K \|v\|_K \|w\|_K.$$

- (c) Deduce that (2) holds.
- (d) Show that  $\mathcal{T}(u, v, w) \in K$ .

**Part 3: Duhamel formula.** Let  $u_0 \in L^d(\mathbb{R}^d)$ . For any  $t \geq 0$  and  $u \in K$ , we define

$$\Phi(u)(t) := e^{t\Delta} u_0 + \mathcal{T}(u, u, u)(t).$$

1. Show that there exists a constant  $C_2$  such that for any  $u \in K$ , we have

$$\|\Phi(u)\|_K \leq C_2 (\|u_0\|_{L^d} + \|u\|_K^3) \quad (3)$$

and that  $\Phi \in K$ .

2. Show that there exists a constant  $C_3$  such that for any  $u$  and  $v$  in  $K$ , we have

$$\|\Phi(u) - \Phi(v)\|_K \leq C_3 (\|u\|_K^2 + \|v\|_K^2 + \|u\|_K \|v\|_K) \|u - v\|_K. \quad (4)$$

(Hint: use Estimate (2))

**Part 4: Fixed point argument** Let  $\varepsilon > 0$ . Assume that

$$\|u_0\|_{L^d} < \varepsilon. \quad (5)$$

and we introduce the set

$$B(2\varepsilon) := \{u \in K \mid \|u\|_K \leq 2\varepsilon\}.$$

1. Show that  $B(2\varepsilon)$  is a complete metric space for the distance induced by the  $K$  norm, namely  $\|\cdot\|_K$ .
2. Show that there exists a constant  $C_4$  which does not depend on  $\varepsilon$ , such that for any  $u$  and  $v$  in  $B(2\varepsilon)$ , we have

$$\|\Phi(u)\|_K \leq C_4(1 + \varepsilon^2)\varepsilon$$

and

$$\|\Phi(u) - \Phi(v)\|_K \leq C_4\varepsilon^2\|u - v\|_K.$$

3. Choose  $\varepsilon > 0$  small enough such that  $\Phi$  is a strict contraction of  $B(2\varepsilon)$ .
4. Deduce that  $\Phi$  has a unique fixed point  $u$ . (*Hint*: use the fixed point theorem.)

**Remark 1.7.1.** *The fixed points of  $\Phi$  are called the mild solutions of the equation*

$$\begin{cases} \partial_t u - \Delta u + u^3 = 0, & \text{in } ]0, +\infty[ \times \mathbb{R}^d, \\ u(0, \cdot) = u_0, & \text{in } \mathbb{R}^d, \end{cases} \quad (6)$$

*for small enough initial data  $u_0$ . We can obtain the existence of mild solution for  $u_0$  large, but only for small times. The same method (so called "Kato method") can be used to show the existence of solution to the Navier-Stokes equation.*