

Sheet 7

Exercise 1.1 (Spectrum of bounded operator)

Let \mathcal{H} be an Hilbert space and $A \in B(\mathcal{H})$.

1. Show that, if $\|A\|_{B(\mathcal{H})} < 1$ is bounded, then $1 + A$ is invertible.
2. Show that $\sigma(A)$ is compact.

Exercise 1.2 (The Lax-Milgram Theorem)

Let \mathcal{H} be a Hilbert space and

$$\alpha : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$$

a sesquilinear form. Assume that

- α is *bounded*: there exists $C > 0$ so that for all $f, g \in \mathcal{H}$

$$|\alpha(f, g)| \leq C\|f\|\|g\|;$$

- α is *coercive*: there exists $a > 0$ so that for all $f \in \mathcal{H}$

$$\alpha(f, f) \geq a\|f\|^2.$$

Prove that:

1. There exists $A \in B(\mathcal{H})$ so that $\alpha(f, g) = \langle Af, g \rangle$;
2. A is bijective with bounded inverse satisfying $\|A^{-1}\| \leq a^{-1}$;
3. $g = A^{-1}f$ is the unique minimiser of

$$g \mapsto \alpha(g, g) - 2\operatorname{Re}\langle f, g \rangle.$$

Exercise 1.3

Let $\lambda > 0$ and $\Phi : \mathbb{C} \rightarrow \mathbb{C}$ such that there exists $L > 0$ satisfying

$$\forall x, y \in \mathbb{C}, \quad |\Phi(x) - \Phi(y)| \leq L|x - y|.$$

1. Let $V \in L^\infty(\mathbb{R}^d)$. Show that, if $\|V\|_{L^\infty}$ is sufficiently small, then there exists a unique weak solution $u \in H^2(\mathbb{R}^d)$ of the equation

$$-\Delta u + \lambda u + V \Phi(u) = 0.$$

2. Let $W \in L^1(\mathbb{R}^d)$. Show that, if $\|W\|_{L^1}$ is sufficiently small, then there exists a unique weak solution $u \in H^2(\mathbb{R}^d)$ of the equation

$$-\Delta u + \lambda u + W * \Phi(u) = 0.$$

Homework (hand in on 30.03.2026).**Exercise 1.4**

Exercise 1.4 from Sheet 6.

Exercise 1.5

Let $V \in L^\infty(\mathbb{R}^d, \mathbb{R})$ be non-negative.

1. Prove that for every $f \in L^2(\mathbb{R}^d)$ and $\lambda > 0$ there exists a unique $u \in H^1(\mathbb{R}^d)$ such that

$$\forall \varphi \in H^1(\mathbb{R}^d) : \langle \nabla u, \nabla \varphi \rangle + \langle (V + \lambda)u, \varphi \rangle = \langle f, \varphi \rangle,$$

that is, there is a unique weak solution to the equation

$$-\Delta u + Vu + \lambda u = f.$$

2. Prove that the weak solution $u \in H^1(\mathbb{R}^d)$ obtained in part 1. is an element of $H^2(\mathbb{R}^d)$.