

Sheet 8

Exercise 1.1 (Cauchy-Lipschitz Theorem)

Let X be a Banach space and $F : X \rightarrow X$ be a Lipschitz map.

1. Show that for any $x_0 \in X$, there exists $x \in C([0, +\infty[; X) \cap C^1(]0, +\infty[; X)$ such that

$$\begin{cases} x' = F(x), & \text{in }]0, +\infty[, \\ x(0) = x_0. \end{cases} \quad (1)$$

2. Show that, for a given initial data $x_0 \in X$, this solution is unique.
3. Let $g \in L^\infty(\mathbb{R}^d)$ and $f_0 \in L^2(\mathbb{R}^d)$. How that the following Cauchy problem have a unique solution

$$\begin{cases} \partial_t f = gf, & \text{in }]0, +\infty[\times \mathbb{R}^d, \\ f(0, \cdot) = f_0, & \text{in } \mathbb{R}^d, \end{cases} \quad (2)$$

where $f : [0, +\infty[\times \mathbb{R}^d \rightarrow \mathbb{C}$.

Exercise 1.2

Let $A \in \mathcal{B}(\mathbb{C}^d) = \mathbb{C}^{d \times d}$ and consider the linear autonomous ODE

$$\frac{du}{dt} = Au(t).$$

Show that

$$\limsup_{t \rightarrow \infty} |u(t)| < \infty$$

holds for all solutions if and only if all eigenvalues of A have non-positive real part and the purely imaginary eigenvalues have equal algebraic and geometric multiplicity.

Give examples where the solution exhibits exponential/polynomical growth.

Exercise 1.3 (Transport equation)

Let $v \in \mathbb{R}$ and $A := v \cdot \nabla$ with $D(A) := H^1(\mathbb{R})$.

1. Show that A is maximal dissipative.
2. Show that for $u_0 \in L^2(\mathbb{R})$

$$(e^{At}u_0)(x) = u_0(x + tv).$$

Exercise 1.4 (Dissipative matrices)

Let $d \in \mathbb{N}$ and $A \in B(\mathbb{C}^d)$ be a $d \times d$ matrix.

1. Assume there exists a unitary $U \in B(\mathbb{C}^d)$ so that UAU^* is diagonal and give a necessary and sufficient condition on $\sigma(A)$ for A to be dissipative.
2. Let $d = 2$ and A be the non-trivial Jordan block

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

Give a necessary and sufficient condition on $\lambda \in \mathbb{C}$ for A to be dissipative.

3. Let A be as in part 2. and $\operatorname{Re} \lambda < 0$. Show that there exists a matrix S such that $B = SAS^{-1}$ is dissipative.

Exercise 1.5 (Ornstein-Uhlenbeck semi-group)

Let us consider $\gamma : x \in \mathbb{R} \mapsto \frac{1}{\sqrt{2\pi}} e^{-\frac{|x|^2}{2}}$ and set

$$L^2(\gamma) := \left\{ f : \mathbb{R} \mapsto \mathbb{R} \text{ measurable} \mid \int_{\mathbb{R}} |f(x)|^2 \gamma(x) dx < +\infty \right\}.$$

The for any $t \geq 0$ and $f \in L^2(\gamma)$, we set

$$(U_t f)(x) := \int_{\mathbb{R}} f(e^{-t}x + \sqrt{1 - e^{-2t}}y) \gamma(y) dy.$$

1. Show that for all $t \geq 0$ and $f \in L^2(\gamma)$, $U_t f$ is well-defined.
2. Show that $(U_t)_{t \geq 0}$ its a contraction semi-group on $L^2(\gamma)$.

★ In the following, we will denote by A is generator of $(U_t)_{t \geq 0}$. ★

3. Let us set

$$\mathcal{A} := \left\{ f \in C^\infty(\mathbb{R}) \mid \forall \alpha \in \mathbb{N}, \exists P \in \mathbb{R}[X] \text{ such that } |f^{(\alpha)}| \leq P \right\}.$$

Show that $\mathcal{A} \subset D(A)$ and that for any $f \in \mathcal{A}$, we have

$$\forall x \in \mathbb{R}, \quad (Af)(x) = \partial_x^2 f(x) - x \partial_x f(x).$$

4. We admit that \mathcal{A} is stable by $(\lambda - A)^{-1}$ for some $\lambda > 0$. Show that $\overline{\mathcal{A}}^{D(A)} = D(A)$ (*Hint*: use that \mathcal{A} is dense in $L^2(\gamma)$).
5. Show that A is self-adjoint.

Exercise 1.6 (The wave equation)

In this exercise we solve the wave equation on \mathbb{R}^d using the Hille Yosida theorem.

The wave equation is

$$\begin{cases} \partial_t^2 u - \Delta u = 0 \\ u(0) = u_0 \\ \partial_t u(0) = v_0. \end{cases} \quad (\text{W})$$

1. Let $\mathcal{H} := H^1(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ and let A be the operator

$$A = \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$$

with domain $D(A) := H^2(\mathbb{R}^d) \oplus H^1(\mathbb{R}^d)$. Show that if $(u, v) \in C^1(\mathbb{R}, \mathcal{H})$ is a solution to the Cauchy problem

$$\begin{cases} \frac{d}{dt}(u, v) = A(u, v) \\ (u, v)(0) = (u_0, v_0) \end{cases} \quad (\text{A})$$

then u solves the wave equation (W).

2. Show that (u, v) solves (A) if and only if $(\tilde{u}, \tilde{v}) = e^{-t}(u, v)$ solves

$$\begin{cases} \frac{d}{dt}(\tilde{u}, \tilde{v}) = (A - 1)(\tilde{u}, \tilde{v}) \\ (\tilde{u}, \tilde{v})(0) = (u_0, v_0). \end{cases}$$

3. Show that $A - 1$ is maximal dissipative.
4. State the existence and uniqueness result for the wave equation implied by 1. and 2. and the Hille-Yosida theorem, specifying the functional space for the solution u .

Homework (hand in on 22.04.2026)**Exercise 1.7 (Magnetic Schrödinger operator)**

Let $B_1 \in C_b^1(\mathbb{R}^d)^d$ such that $\operatorname{div}(B_1) = 0$. We consider the magnetic Schrödinger operator

$$A := (-i\nabla + B_1)^2, \quad D(A) := H^2(\mathbb{R}^d).$$

Show that $(A, D(A))$ is maximal dissipatif.